2025 TAIPEI

Geometric Integration for Neural Control Variates

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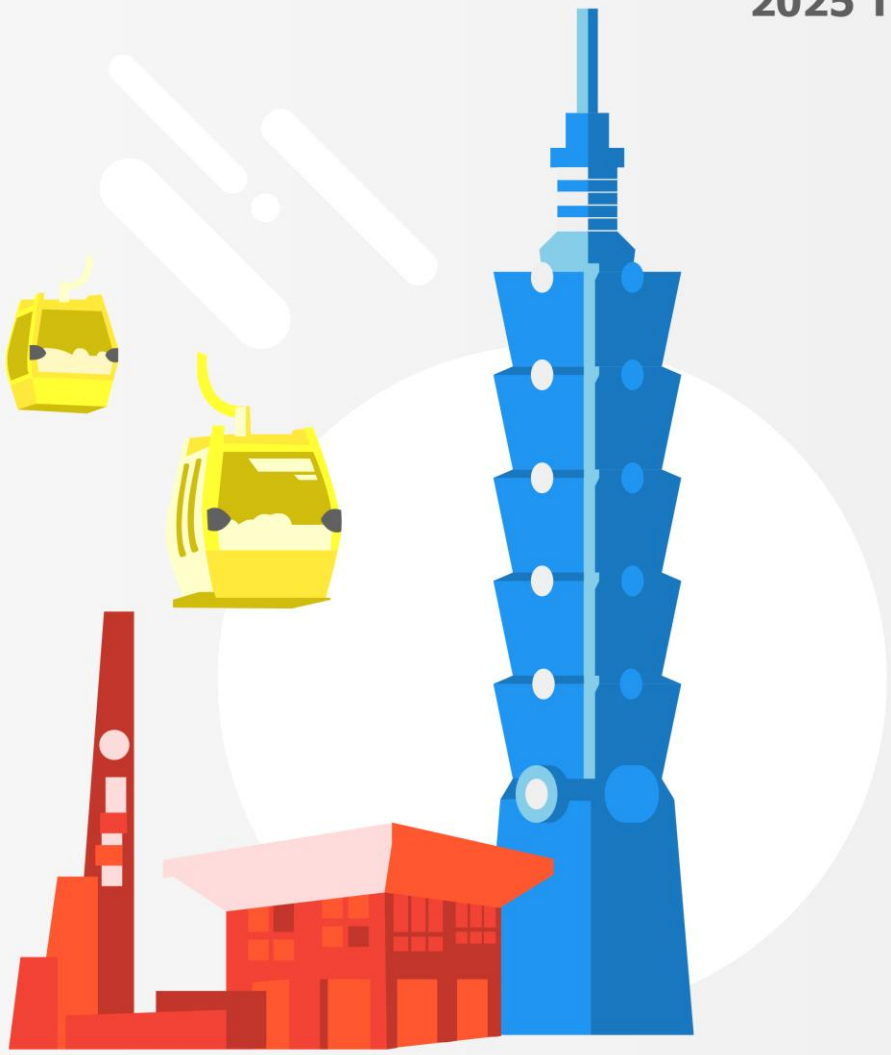
AMD



Computer Graphics Problems

Many problems in computer graphics can be formulated as integral equations

- Geometric processing (e.g., Laplace equation)
- Light transport (e.g., the rendering equation)

$$L(\mathbf{x}, \boldsymbol{\omega}_o) = L_e(\mathbf{x}, \boldsymbol{\omega}_o) + \int_{\boldsymbol{\omega}_i \in \Omega} f_r(\mathbf{x}, \boldsymbol{\omega}_i, \boldsymbol{\omega}_o) L(\mathbf{x}, \boldsymbol{\omega}_i) \cos\theta_r \, \mathrm{d}\boldsymbol{\omega}_i$$

No analytic solution in general → numerical integration

All problems can be formulated as a general integral on some domain

$$F = \int_{\mathcal{D}} f(x) \, \mathrm{d}x$$

Monte Carlo (MC) Integration

Numerical integration widely used in computer graphics

- High-dimensional problems
- Robust to discontinuities
- Converges to the true solution
  - Many samples needed to surpass the variance (noise)

$$F = \int_{\mathcal{D}} f(x)\,\mathrm{d}x \approx \frac{1}{n}\sum_{i=1}^{n}\frac{f(x_i)}{p(x_i)}$$

Variance reduction techniques

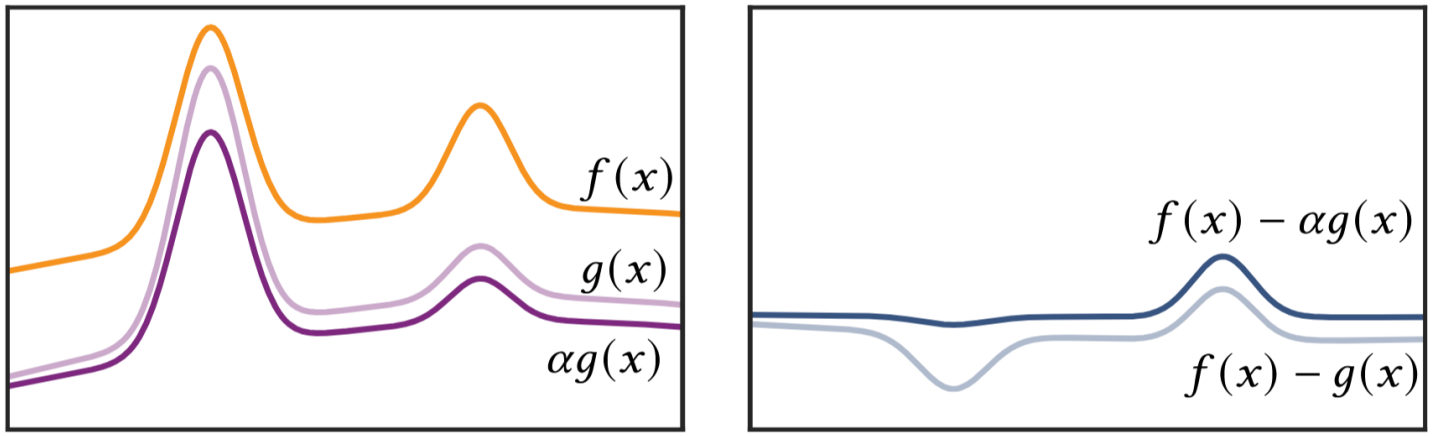
- Importance sampling
- **Control variates**

Monte Carlo with Control Variates

Approximate the integrand and integrate only the residual integral

- We assume $\alpha = 1$ from now on

$$F = \int_{\mathcal{D}} f(x)\,\mathrm{d}x = \alpha G + \int_{\mathcal{D}} f(x) - \alpha g(x)\,\mathrm{d}x \qquad G = \int_{\mathcal{D}} g(x)\,\mathrm{d}x$$



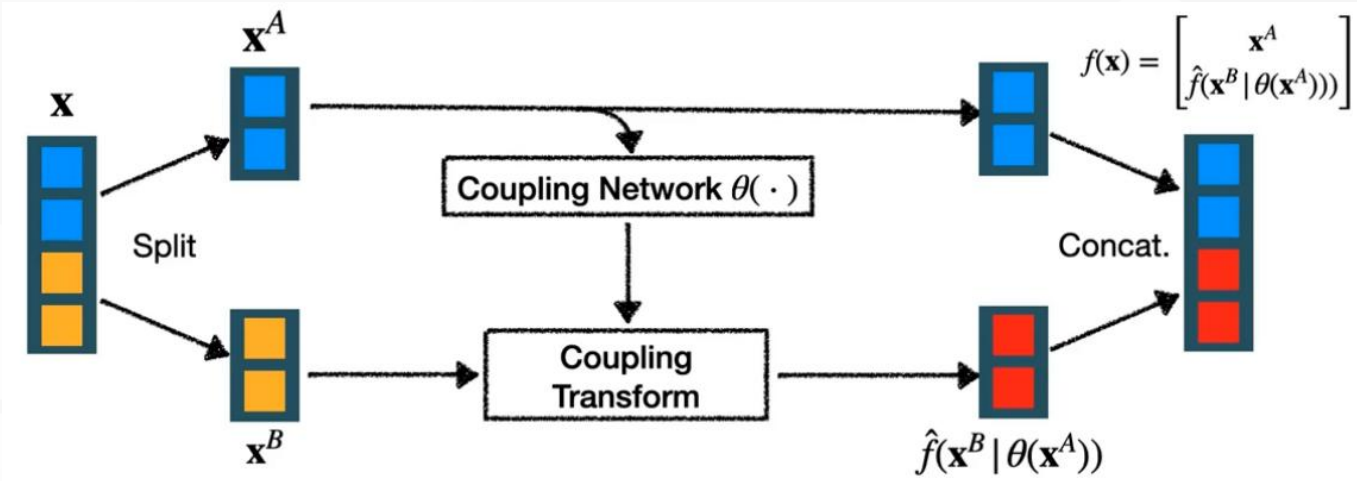
Courtesy of [Muller et al. 2020]

Neural networks are universal approximators. Why not use them as control variates?

Integrating neural network is challenging

Prior Work - Normalizing Flows [Muller et al. 2020]

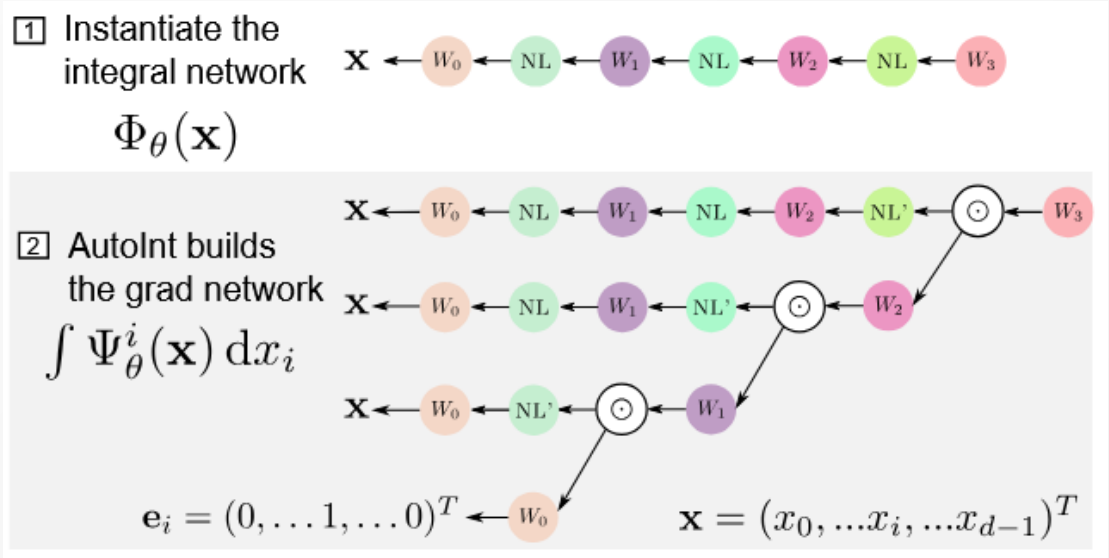
- Normalized probability distribution
  - Scaled by another neural network
  - Scale trivially corresponds to *G*
- A chain of coupling layers (small neural networks)
  - Invertibility limits expressiveness
  - We do not need an inverse mapping
- Does not work well in low dimensions



Courtesy of [Brubaker et al. 2020]

Prior Work – Automatic Integration [Lindell et al. 2021, Li et al. 2024]

- Neural network represents antiderivative
  - A pair of networks: grad and integral networks
  - Grad network is constructed based on the integral network
- Requires autodiff and higher order derivatives
  - The integral network must be differentiable enough
  - Non-zero *n*-th order derivatives for *n*-dimensions
- Grad network can be huge
  - Difficult training
  - Size scales with the number of dimensions



Courtesy of [Lindell et al. 2021]

Contribution

Control variates using the MLP with *piece-wise linear activation functions* (ReLU)

- Analytic integration of the MLP as a geometric problem
- Employing tools from computational geometry to solve it in 2D
- Light transport applications

An MLP with piece-wise linear activation functions represents a piece-wise linear (affine) function

MLP = Multilayer perceptron
ReLU = Rectified linear unit

Geometric Integration

- Split integration domain into disjoint subdomains
- Such that the function is affine on the subdomain
- Integrate the corresponding affine function on each subdomain

$$G = \int_{\mathcal{D}} g(x)\,\mathrm{d}x = \sum_{i=1}^{n} \int_{\mathcal{D}_i} g(x)\,\mathrm{d}x = \sum_{i=1}^{n} \int_{\mathcal{D}_i} g_i(x)\,\mathrm{d}x \qquad \bigcup_{i=1}^{n} \mathcal{D}_i = \mathcal{D} \qquad \bigcap_{i=1}^{n} = \emptyset$$

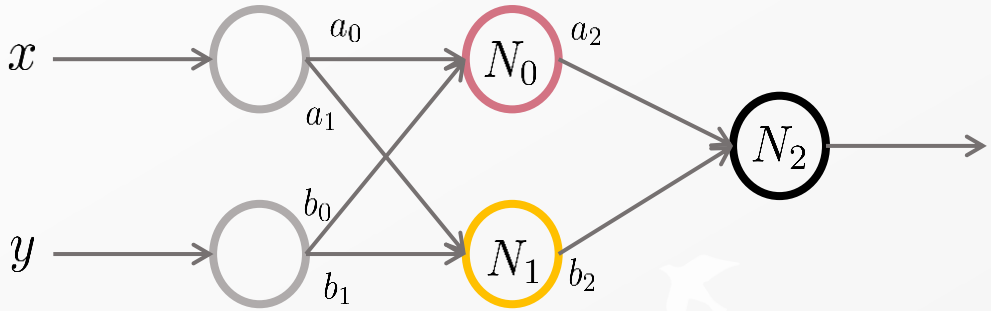
- Subdomains are convex regions
  - Intersection of half-planes
  - Half-planes correspond to inequalities in the activation functions
- The resulting function is always affine
  - Affine functions are closed under composition
  - Composition of linear layers and piece-wise linear activation functions

Finding the subdomains is the challenging part

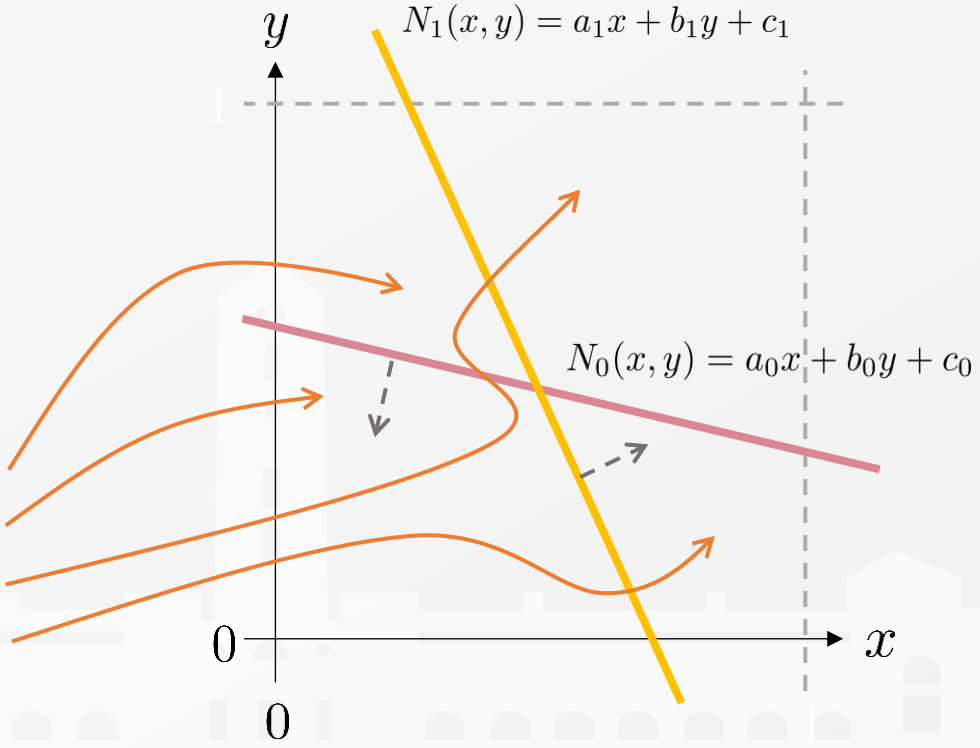
Geometric Integration – Example

MLP with one hidden layer and ReLU activation functions

- Integration domain is a unit square
- Each neuron corresponds to a half-plane (an oriented line)
- The resulting function depends on the neuron activations



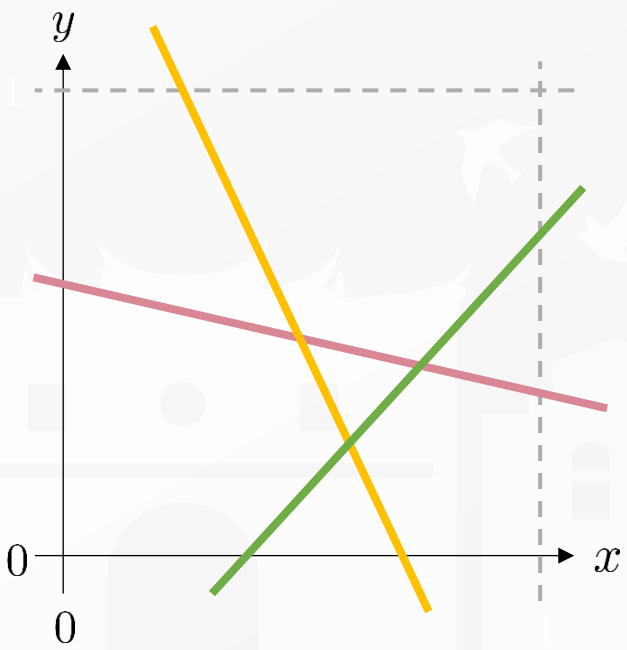
$$N_2(x,y) = \begin{cases} c_2 & \text{if } N_0(x,y) < 0 \text{ and } N_1(x,y) < 0 \\ a_2(a_0x + b_0y + c_0) + c_2 & \text{if } N_0(x,y) \geq 0 \text{ and } N_1(x,y) < 0 \\ b_2(a_1x + b_1y + c_1) + c_2 & \text{if } N_0(x,y) < 0 \text{ and } N_1(x,y) \geq 0 \\ a_2(a_0x + b_0y + c_0) + b_2(a_1x + b_1y + c_1) + c_2 & \text{if } N_0(x,y) \geq 0 \text{ and } N_1(x,y) \geq 0 \end{cases}$$

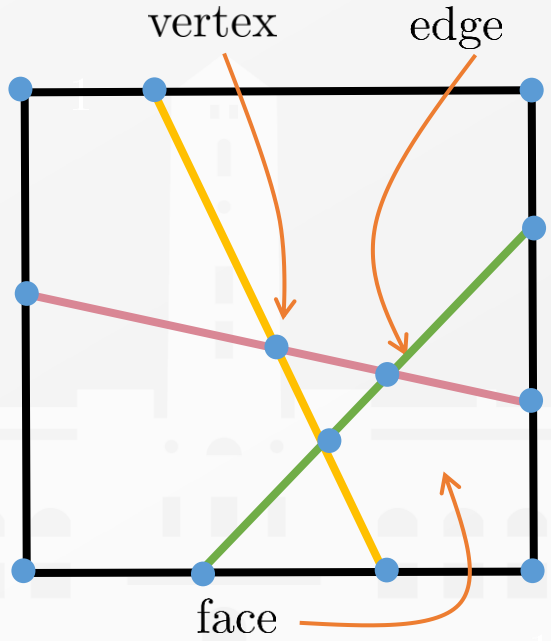


We need to think in terms of affine functions instead of numbers!

Arrangement of Lines

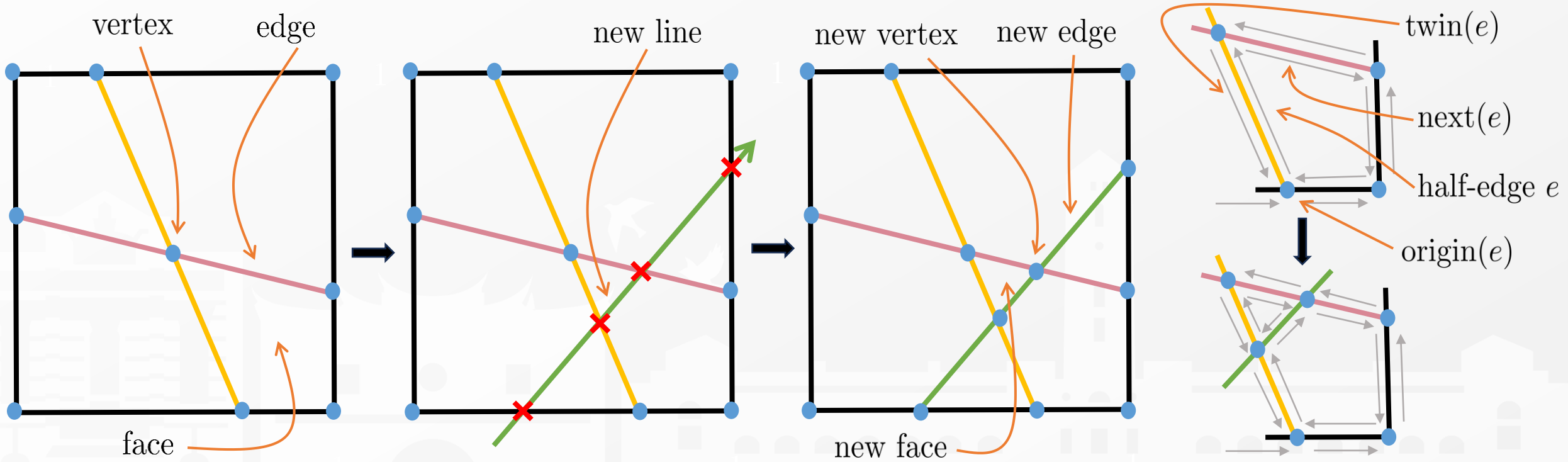
- Well-studied problem in computational geometry
- Find a planar subdivision induced by given a set of lines in 2D
  - Vertices – points where lines meet (no more than $\binom{N}{2}$)
  - Edges – line segments between points (no more than N^2)
  - Faces – convex regions containing no lines (no more than $\binom{N}{2} + N + 1$)
  - N is the number of lines (the number of neurons in the a hidden layer)





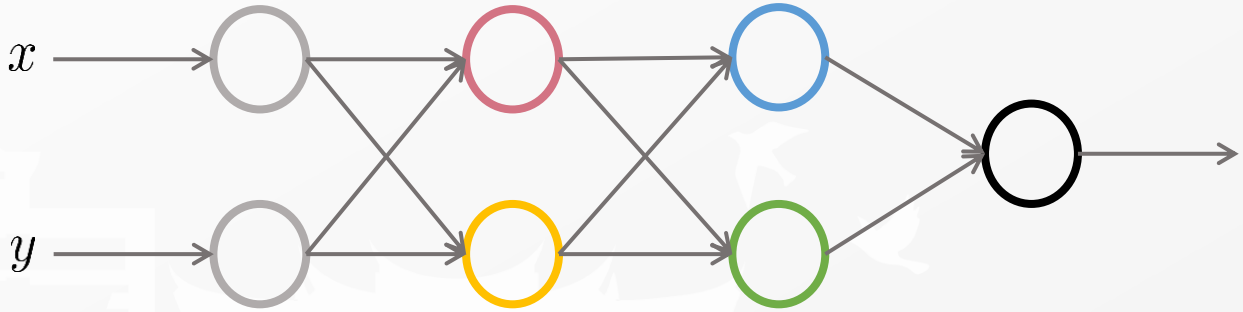
Doubly-Connected Edge List (DCEL)

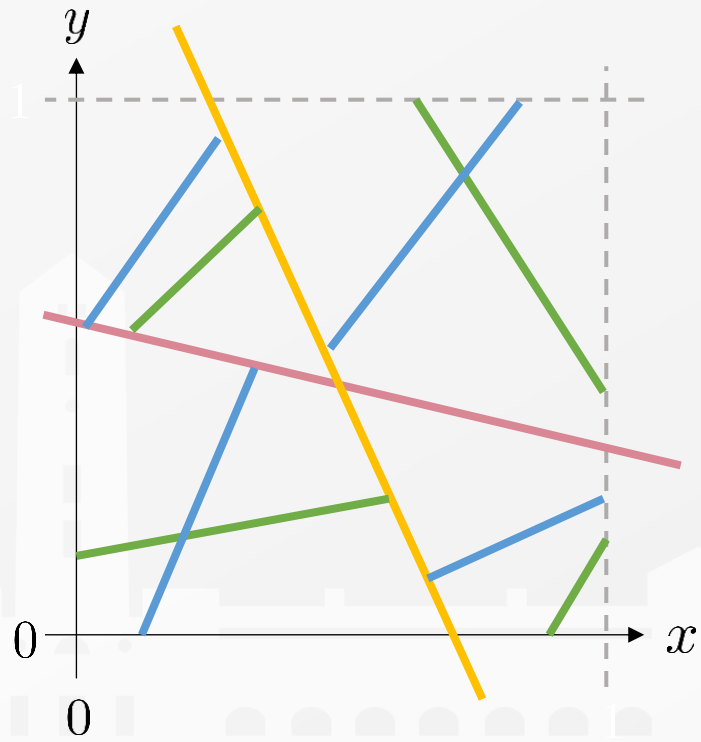
- A data structure for storing a planar subdivision
- Topological information
- Incremental construction – “2D ray tracing”



Multiple Hidden Layers

- The half-planes are mixed based on the activation from the previous layer
- Different activations result in different half-planes in each face
- Recursive processing by traversing “very wide tree”
  - The number of neurons in a layer corresponds to the branching factor





We can relatively easily integrate the corresponding affine function once the subdomain is determined

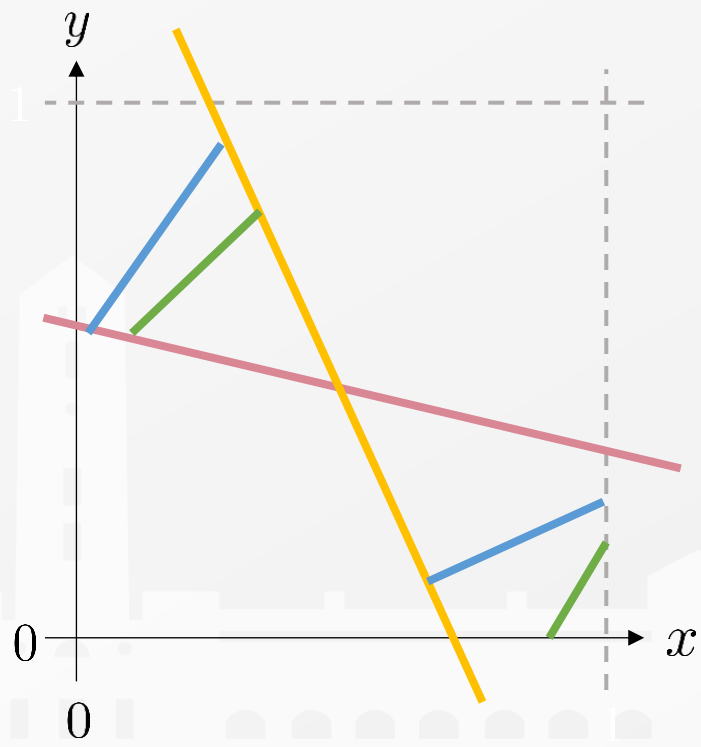
Please, see the details in the paper

Traversal Algorithm

- The goal is to visit all bottom-most “leaf” faces
  - Simple stackless logic
- No need to store the whole subdivision
  - Once the face at any level is processed
  - The allocated DCELs can be reused for other faces
- Allocating only one DCEL per layer

Missing details

- The lines are oriented (half-planes)
- Track the resulting affine function (for the integration)

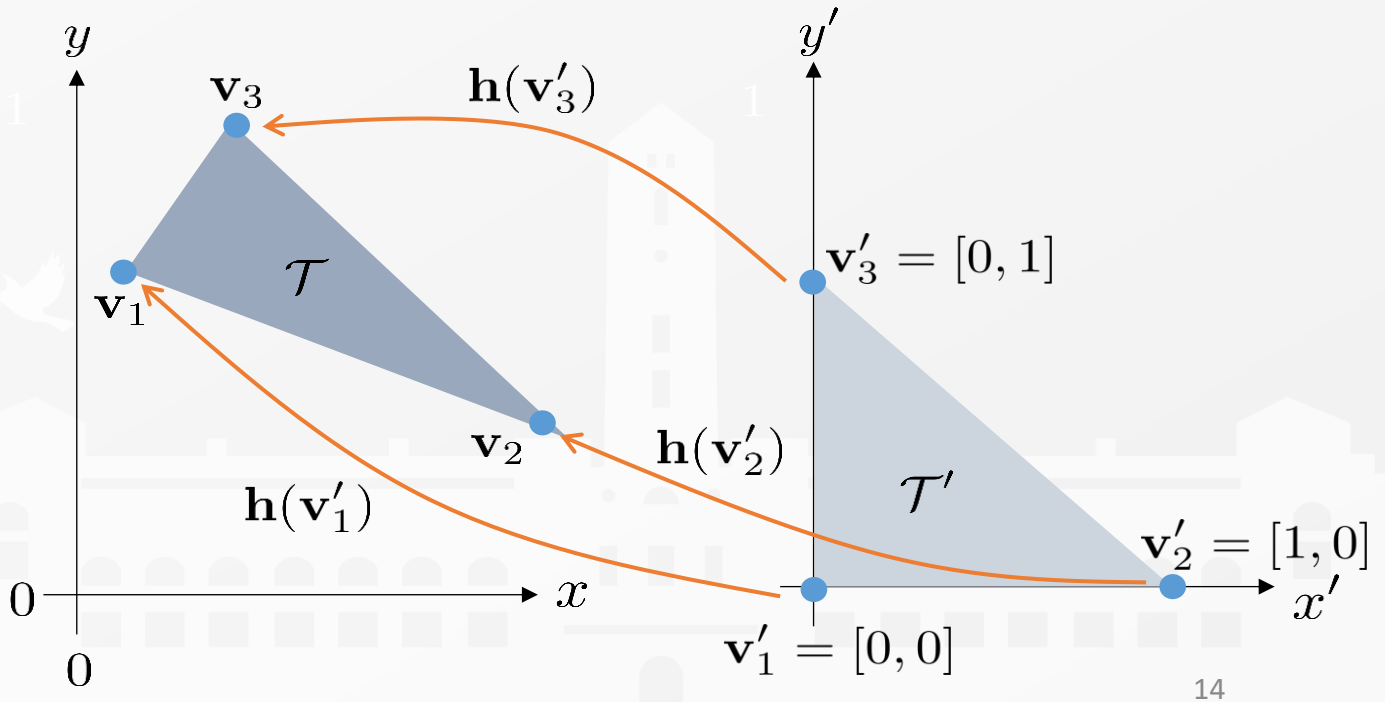


Triangle Integration

- Convex regions can be trivially triangulated
- One of the variable needs to be independent
  - Iterative (nested) integrals computed as 2x 1D integrals
- Integrating over a unit triangle
  - Mapping back to the original one
  - The mapping is an affine function

$$G = \sum_{i=1}^{n} \int_{\mathcal{D}_i} g_i(x, y)\,\mathrm{d}y\,\mathrm{d}x = \sum_{i=1}^{n} \sum_{j=1}^{m_i} \int_{\mathcal{T}_{i,j}} a_i x + b_i y + c_i\,\mathrm{d}y\,\mathrm{d}x$$

$$2A_{\mathcal{T}} \int_0^1 \int_0^{1-x'} a'x' + b'y' + c'\,\mathrm{d}y'\,\mathrm{d}x' = \frac{A_{\mathcal{T}}}{3}(a' + b' + 3c')$$

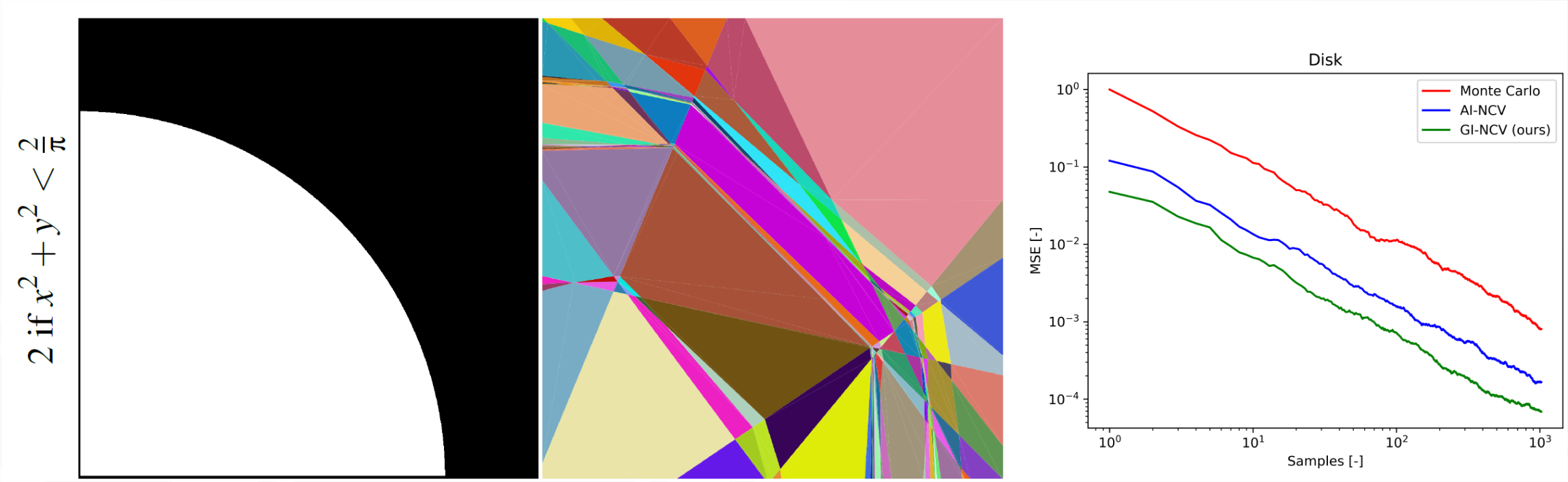


Applications: Analytic Functions

- Proof-of-concept implementation in PyTorch
- MLP with 2 hidden layers with 32 neurons each with ReLU
- Automatic integration [Li et al. 2024] as a reference method
  - Relying on the autograd in PyTorch
  - Sigmoid instead of ReLU

Disk Function

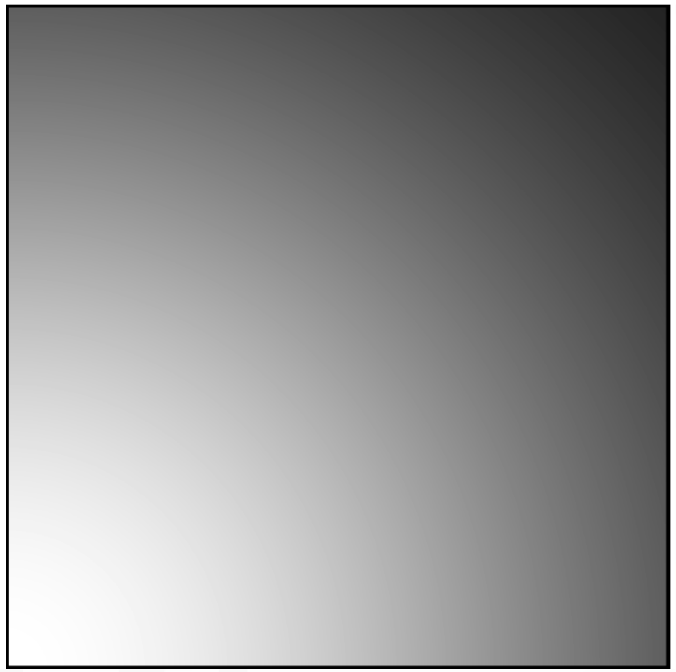
- Both methods are competitive
- The main difference is due to a different activation function

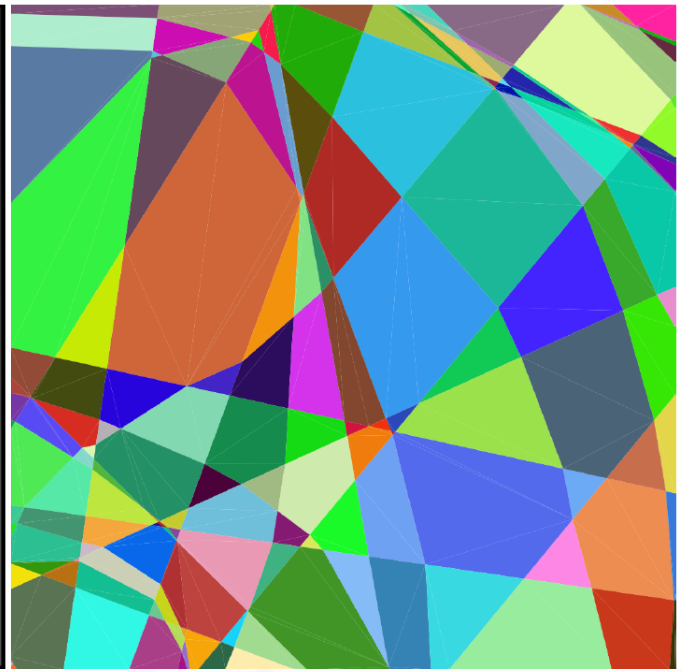


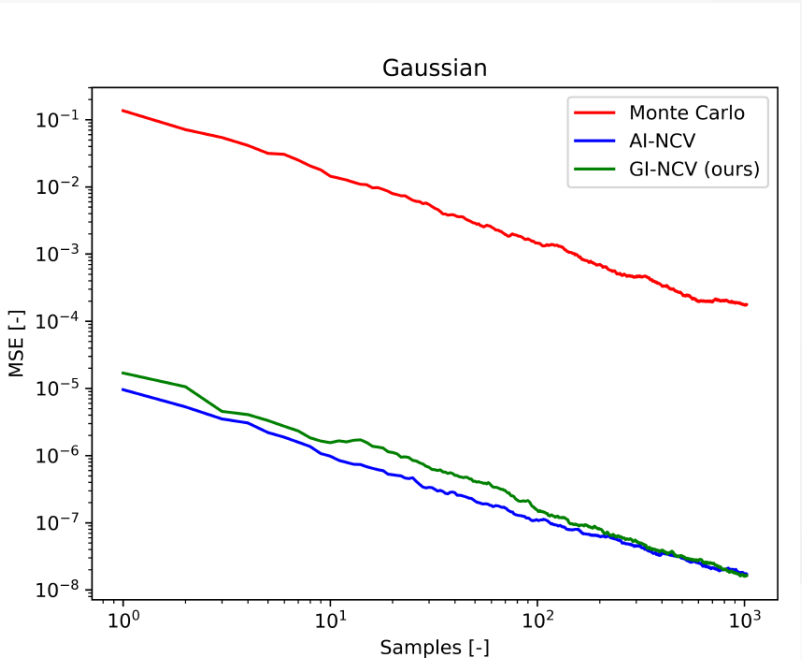
Gaussian Function

- Both methods are competitive
- The main difference is due to a different activation function

$$\frac{4}{\pi \mathrm{erf}^2(1)} \exp\left(-x^2 - y^2\right)$$







Applications: Light Transport

- Solving many integrals (e.g., one per pixel)
- A single neural network $g(x, y \mid \phi)$
  - 2 hidden layers with 32 neurons each
- Arbitrary additional inputs ϕ
  - Fixed w.r.t integration
  - Arbitrary encoding
- Integrating a 2D slice of a higher dimensional function

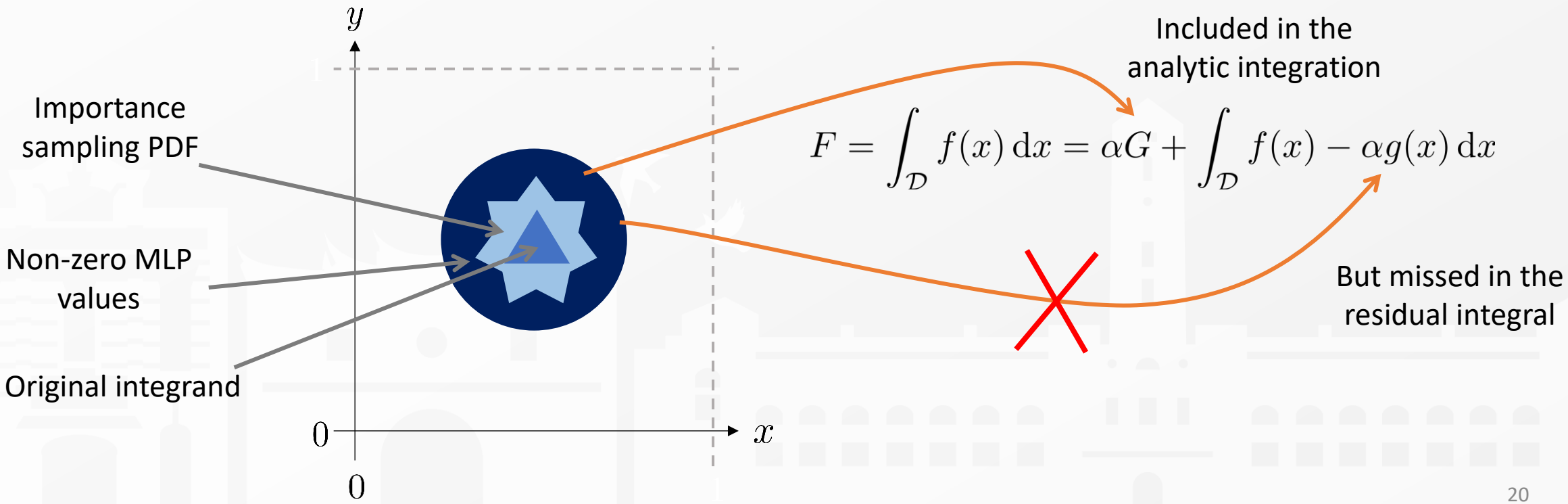
Parameter	Symbol	Encoding	Input dim.	Output dim.
Area light sample	(x, y)	Identity	2	2
Incoming direction	ω_i	Local hemisph. coords.	2	2
Position	$\mathbf{x}$	Hashgrid	3 (2)	2×4
Outgoing direction	ω_o	Sph. coords. / one-blob	2	2×4
Surface normal	$\mathbf{n}(\mathbf{x})$	Sph. coords. / one-blob	2	2×4
Diffuse albedo	$\sigma(\mathbf{x})$	Identity	3	3
Surface roughness	$\rho(\mathbf{x})$	Identity	1	1

Implementation Details

- Pre-train → Geometric Integration → Integrate residual integral using MC
- In-house HIP renderer & MLP framework
  - HIPRT 2.5 for ray tracing
  - Adam optimizer
  - Weight & biases in FP32
  - Exponential moving average for weights & biases
- Geometric integration as a separate kernel
  - Persistent threads to reduce memory requirements
- One training sample per pixel
- AMD Radeon PRO W7900 (48GB)

Control Variates and Importance Sampling

- Analytic integration domain must match integration of domain of the residual integral
- Probability density might be zero where the original integrand is zero
- MLP might provide arbitrary values, which are included in the analytic integration
- But these samples might be missed in the residual integral due to importance sampling



Ambient Occlusion

- Integration over upper hemisphere

$$AO(\mathbf{x} \mid r) = \int_{\omega_i \in \Omega} \frac{V(\mathbf{x}, \omega_i \mid r)(\mathbf{n}(\mathbf{x}) \cdot \omega_i)}{\pi} \, \mathrm{d}\omega_i$$

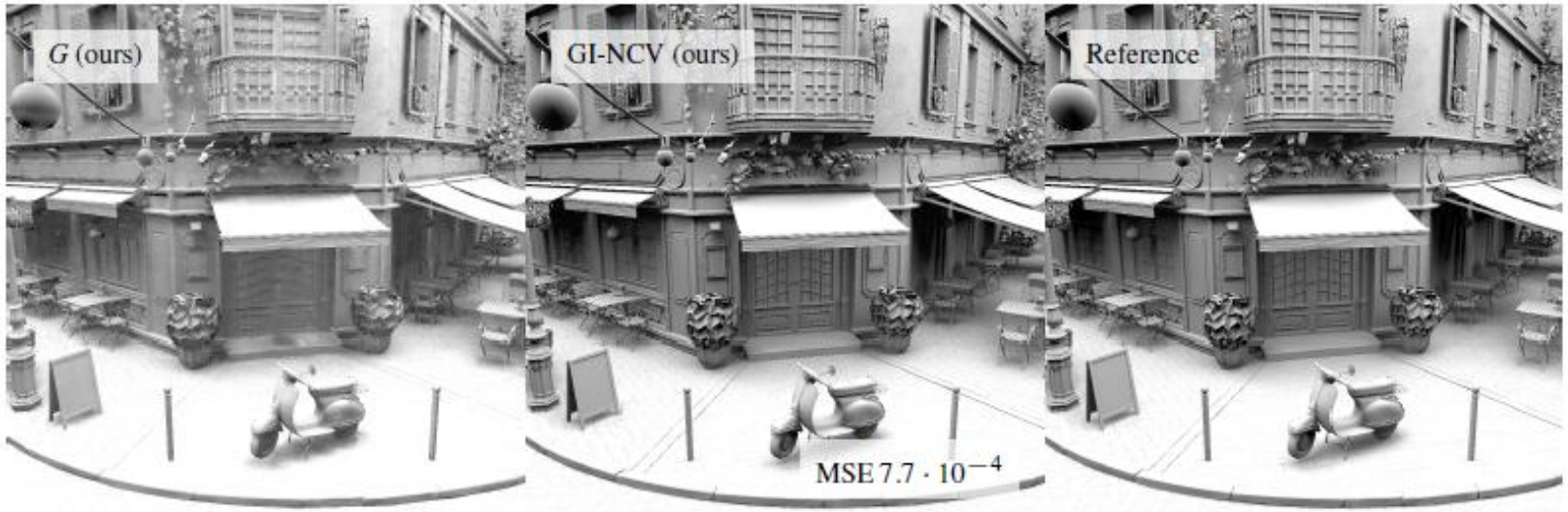
- Approximating integrand

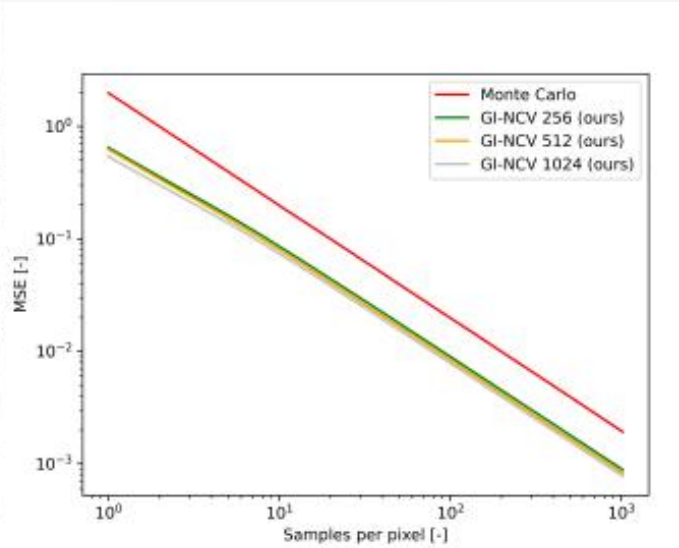
$$g(\omega_i \mid \mathbf{x}, \mathbf{n}(\mathbf{x})) \approx f(\mathbf{x}, \omega_i \mid r) = \frac{V(\mathbf{x}, \omega_i \mid r)(\mathbf{n}(\mathbf{x}) \cdot \omega_i)}{\pi}$$

- Direction in local hemispherical coordinates
  - Transform to the local space
  - Hemi-spherical coordinates (height & azimuth)

Ambient Occlusion - Bistro

Comparison to vanilla Monte Carlo using different number of training iterations





Direct Lighting

- Integration over an area light source

$$L_o^{direct}(\mathbf{x}, \omega_o) = \int_{\mathbf{y} \in A} f_r(\mathbf{x}, \omega_{\mathbf{x}\to\mathbf{y}}, \omega_o) L_e(\mathbf{y}, \omega_{\mathbf{y}\to\mathbf{x}}) G(\mathbf{x}, \mathbf{y}) V(\mathbf{x}, \mathbf{y}) \, \mathrm{d}\mathbf{y}$$

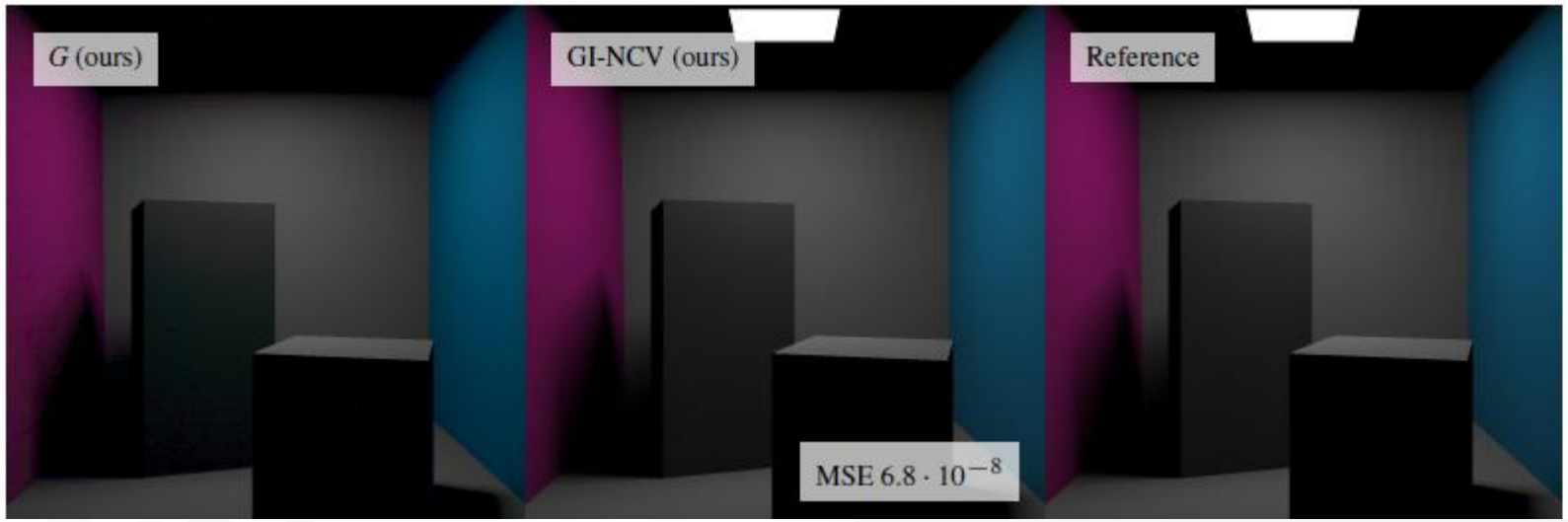
- Approximating integrand

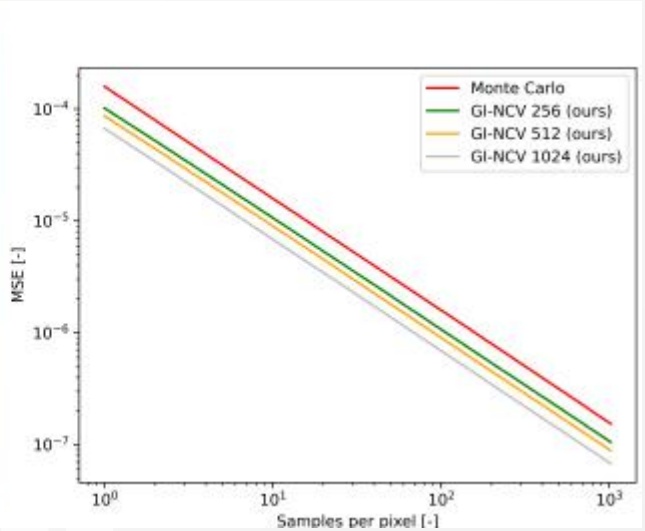
$$\mathbf{g}(x, y \mid \mathbf{x}, \omega_o, \mathbf{n}(\mathbf{x}), \sigma(\mathbf{x}), \rho(\mathbf{x})) \approx \mathbf{f}(\mathbf{x}, \mathbf{v} + x\mathbf{e}_1 + y\mathbf{e}_2, \omega_o) = \mathbf{f}(\mathbf{x}, \mathbf{y}, \omega_o)$$

$$\mathbf{f}(\mathbf{x}, \mathbf{y}, \omega_o) = f_r(\mathbf{x}, \omega_{\mathbf{x}\to\mathbf{y}}, \omega_o) L_e(\mathbf{y}, \omega_{\mathbf{y}\to\mathbf{x}}) G(\mathbf{x}, \mathbf{y}) V(\mathbf{x}, \mathbf{y}),$$

Direct Lighting – Cornell Box

Comparison to vanilla Monte Carlo using different number of training iterations





Global Illumination

- Integration over upper hemisphere

$$L_o(\mathbf{x}, \omega_o) = L_e(\mathbf{x}, \omega_o) + \int_{\omega_i \in \Omega} f_r(\mathbf{x}, \omega_i, \omega_o) L_i(\mathbf{x}, \omega_i)(\mathbf{n}(\mathbf{x}) \cdot \omega_i) \, \mathrm{d}\omega_i$$

- Approximating integrand

$$\mathbf{g}(\omega_i \mid \mathbf{x}, \omega_o, \mathbf{n}(\mathbf{x}), \sigma(\mathbf{x}), \rho(\mathbf{x})) \approx \mathbf{f}(\mathbf{x}, \omega_i, \omega_o) = f_r(\mathbf{x}, \omega_i, \omega_o) L_i(\mathbf{x}, \omega_i)(\mathbf{n}(\mathbf{x}) \cdot \omega_i)$$

Indirect Lighting – Bistro

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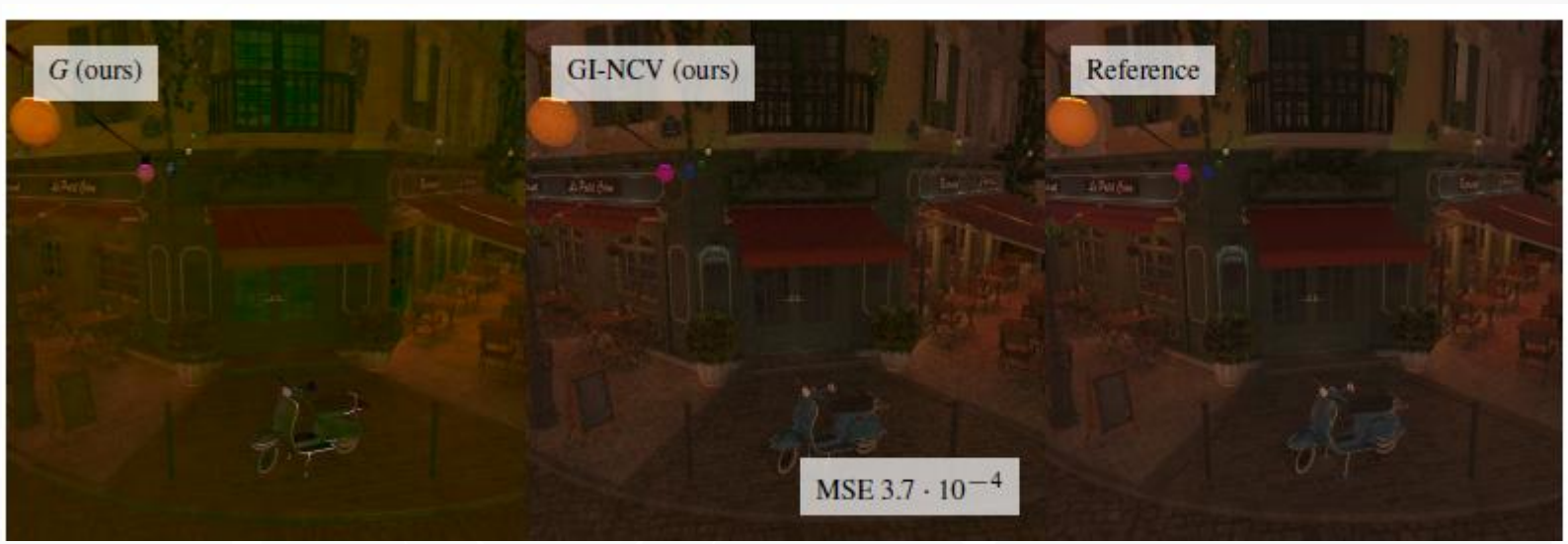
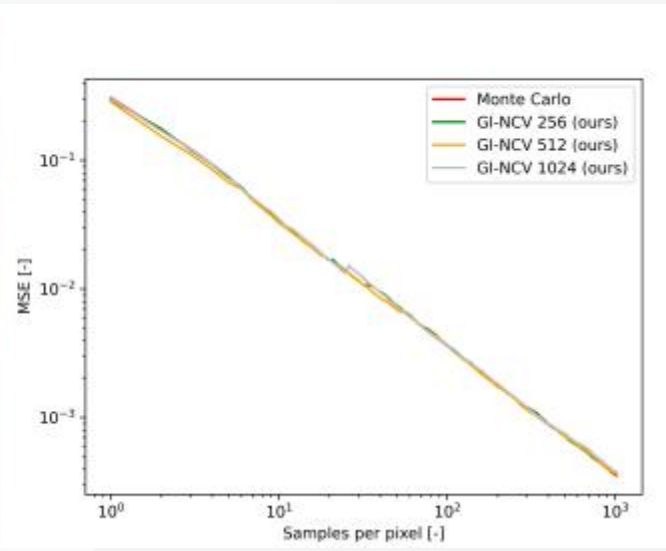


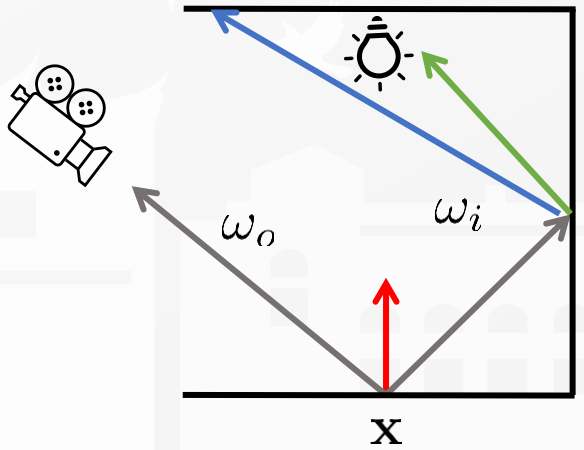


Even though the MLP approximate radiance field well,
control variates does not bring any improvement due to noisy estimates

Global Illumination – Noisy Estimates

- Approximating 7D function (position and two directions)
  - We need to train the whole integrand compared to *radiance caching*
- Notice that the ground truth data are not available
  - Incidence radiance is a nested integral itself
  - MLP can still learn the function from one-sample noisy estimates
  - Thanks to Adam optimizer and exponential moving average (tracking history ~ averaging)

$$\mathbf{g}(\omega_i \mid \mathbf{x}, \omega_o, \mathbf{n}(\mathbf{x}), \sigma(\mathbf{x}), \rho(\mathbf{x})) \approx \mathbf{f}(\mathbf{x}, \omega_i, \omega_o) = f_r(\mathbf{x}, \omega_i, \omega_o) L_i(\mathbf{x}, \omega_i)(\mathbf{n}(\mathbf{x}) \cdot \omega_i)$$



The same argument but different function values

Noisy Estimates - Experiment

Integrating a simple function

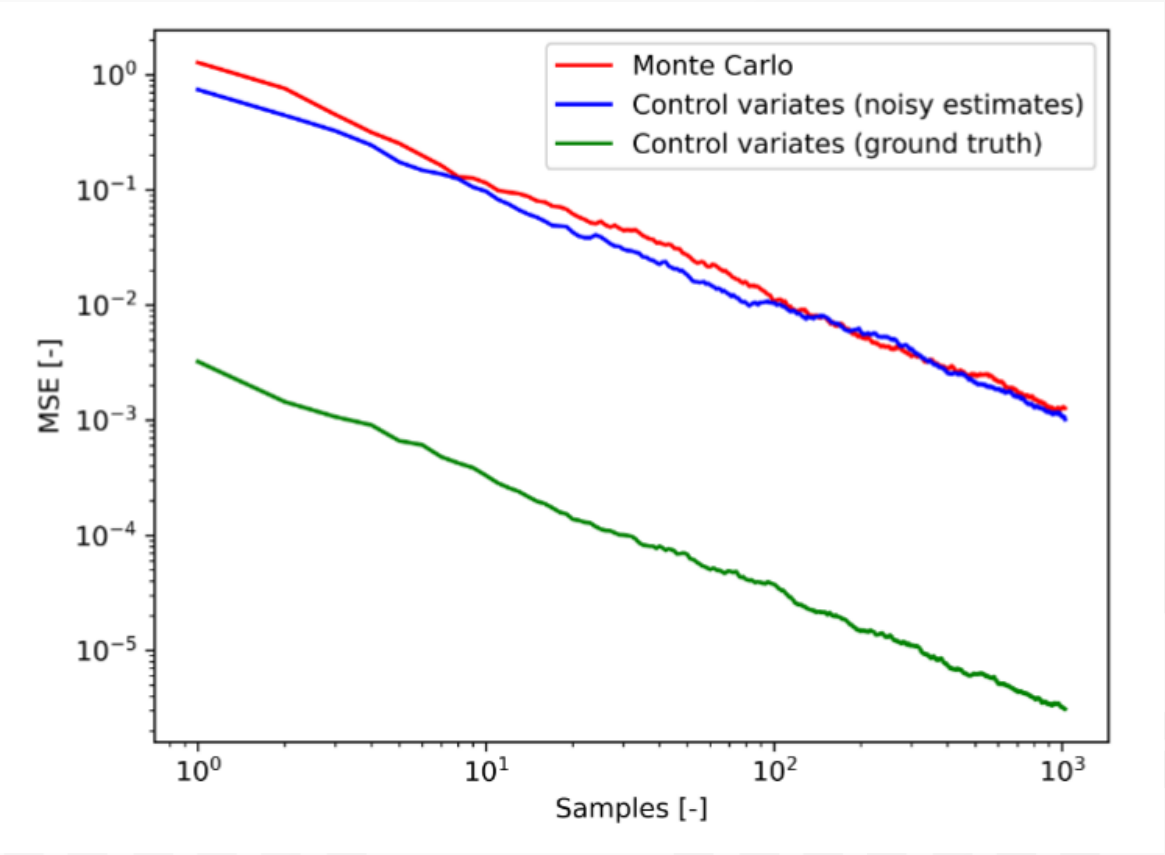
$$f(x) = \int_0^1 \int_0^1 8xyz \, \mathrm{d}y \, \mathrm{d}z = 2x$$

Control variate (almost perfect)

$$g(x) = 0.9 \cdot f(x)$$

Noisy estimates vs. ground truth values

$$\bar{f}(x) = 8xy_iz_i \approx f(x) = 2x$$



Discussion and Limitations

- Same as automatic integration limited to low dimensions (2D)
  - Cannot use full path tracing (with Russian roulette)
  - Possible to extend to higher dimensions in theory but it would be prohibitively expensive
- Same as other control variates methods difficult to use importance sampling
  - Analytic integration covers the whole integration domain (i.e., the unit square)
- Analytic integration is expensive (both in time and space)
  - Amortize the cost through many samples
  - Cannot use in real-time rendering
- Cannot use non-linear encoding for the integration domain
  - To keep the lines straight (not to become curvy)

	Monte Carlo	GI-NCV (ours)
SPP [-]	32768	1024 + 18432
MSE [-]	$4.54 \cdot 10^{-5}$	$4.53 \cdot 10^{-5}$
Render time [s]	136	78
Train time [s]	-	33
Integration time [s]	-	10
Inference time [s]	-	34
Total time [s]	136	155

Conclusion

- Analytic integration of MLP with piece-wise linear act. functions
  - Viable alternative to the automatic integration
- Integration formulated as geometric problem in 2D
  - Practical algorithm relying on tools from computational geometry
  - Looking at MLPs from a different perspective
- Applications in light transport
  - Reducing variance using control variates
  - Many limitations for practical use

Future work

- Importance sampling using the subdivision

Thank you for your attention!

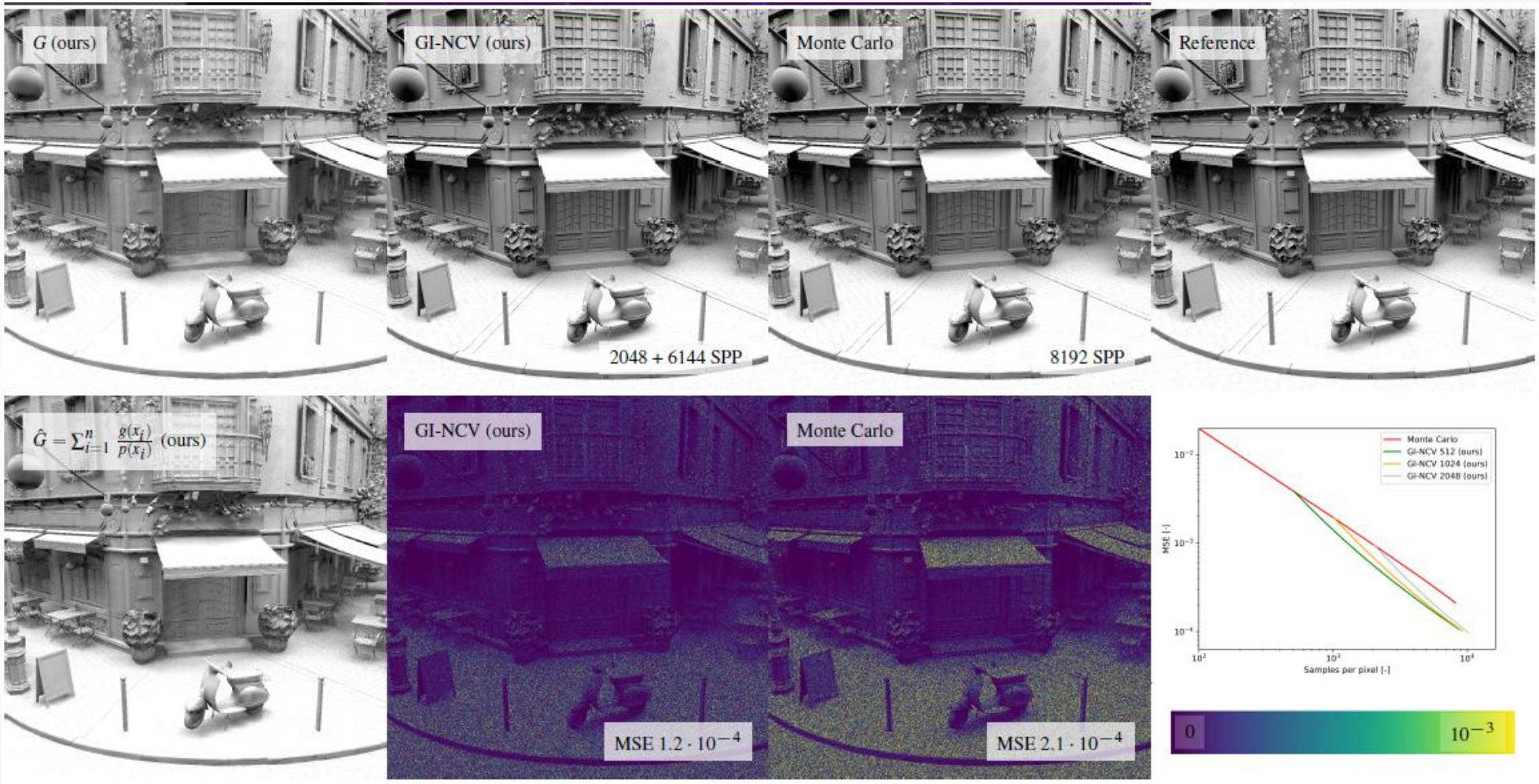
Backup Slides

The Number of Faces

- The particular number depends on the weights & biases
- Some lines might be completely outside the unit square
  - An does not introduce new faces

Ambient Occlusion - Bistro

Using the same samples for training and integration (correlated samples)



Direct Lighting – Cornell Box

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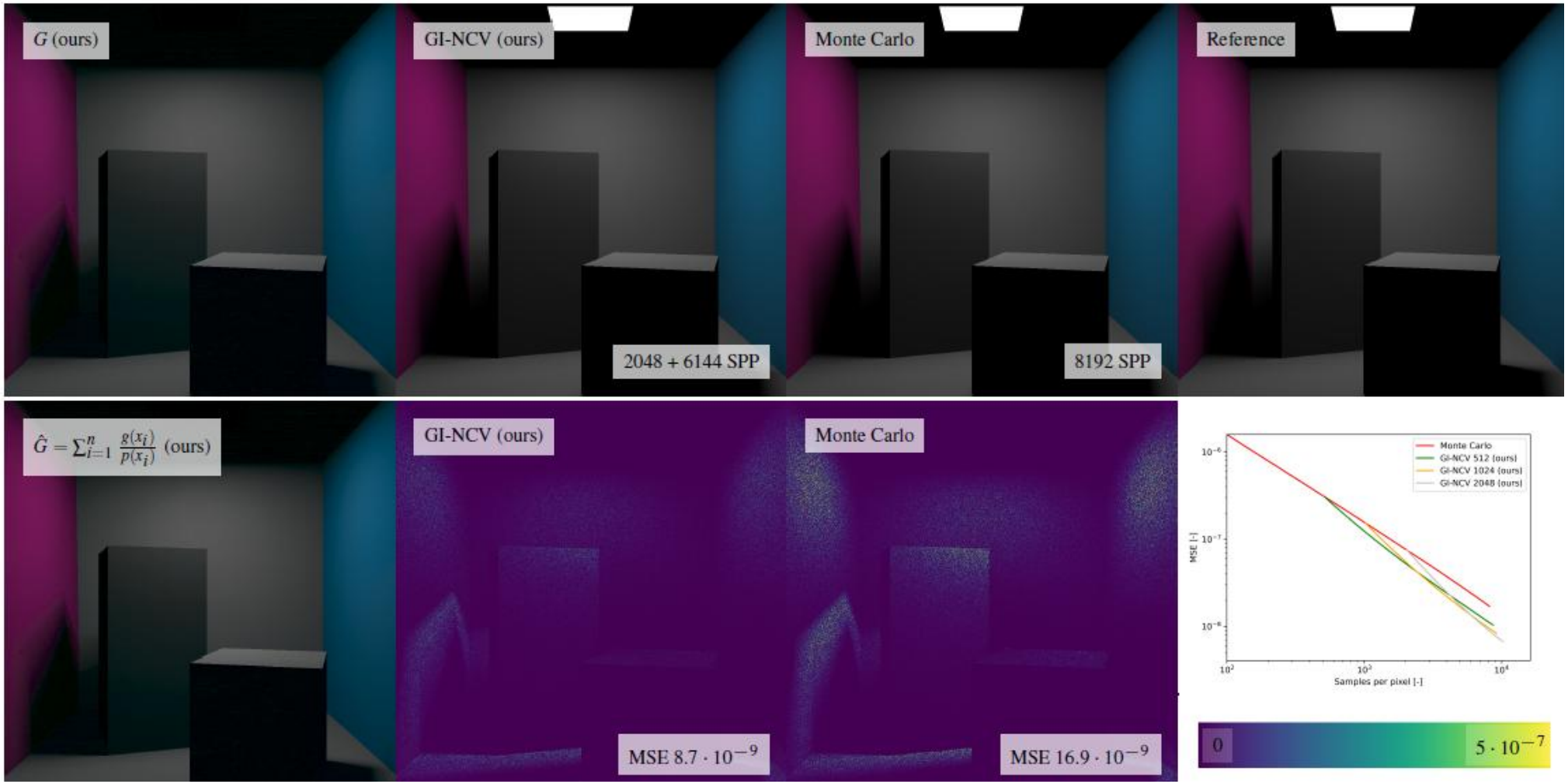

Correlated Samples

$$\langle I \rangle = \frac{m}{m+n}\langle I_{MC} \rangle + \frac{n}{m+n}\langle I_{CV} \rangle,$$

$$\langle I_{MC} \rangle = \frac{1}{m}\sum_{i=1}^{m}\frac{f(x_i)}{p(x_i)}$$

$$\langle I_{CV} \rangle = G + \frac{1}{n}\sum_{i=1}^{n}\frac{f(x_{m+i}) - g(x_{m+i})}{p(x_{m+i})}$$

$$\begin{aligned}
\mathbb{E}[\langle I \rangle] &= \mathbb{E}\left[\frac{m}{m+n}\langle I_{MC} \rangle\right] + \mathbb{E}\left[\frac{n}{m+n}\langle I_{CV} \rangle\right], \\
&= \frac{mF}{m+n} + \frac{n}{m+n}\mathbb{E}\left[G + \frac{1}{n}\sum_{i=1}^{n}\frac{f(x_{m+i}) - g(x_{m+i})}{p(x_{m+i})}\right], \\
&= \frac{mF}{m+n} + \frac{n}{m+n}\left(G + \mathbb{E}\left[\frac{1}{n}\sum_{i=1}^{n}\frac{f(x_{m+i}) - g(x_{m+i})}{p(x_{m+i})}\right]\right), \\
&= \frac{mF}{m+n} + \frac{n}{m+n}(G + F - G) = \frac{(m+n)F}{m+n} = F.
\end{aligned}$$